Collision-induced timing shifts in dispersion-managed soliton systems

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The frequency and timing shifts associated with dispersion-managed solitons in a wavelength-division multiplexed system are computed by the numerically efficient Poisson sum technique. Analytical formulas are attainable by use of this approach with a Gaussian approximation for the soliton. The results are favorably compared with known results for the frequency shift. The method also applies to quasi-linear return-to-zero transmission formats.

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It is well known that wavelength-division multiplexing (WDM) has the potential to increase vastly the data transmission capacity in optical fiber communications environments. However, WDM creates certain penalties, such as frequency and timing shifts due to pulse collisions, in the transmission process. In classical soliton systems it has been shown that these shifts can be quite significant.[1,2] The wide interest in strong dispersion management has motivated several studies of this issue for dispersion-managed (DM) soliton systems as well.[3–9] These studies have relied mainly on numerical simulations or have restricted the range of allowed frequency separations. In this Letter we consider collision-induced frequency and timing shifts due to DM soliton interactions in a two channel system. A new method for computing the frequency and timing shifts, based on the Poisson sum formula, is introduced. By use of a Gaussian approximation for the DM solitons, analytical formulas can be obtained. The technique is straightforward to apply and numerically efficient.

We show that, with these simple Gaussian approximations of the DM soliton, the Poisson sum formula yields the same results for the frequency shifts as those obtained via variational approaches combined with numerical evaluation of the associated integrals.[3] We then extend the Poisson sum method to find the timing shifts. Even though the parameter values used in this Letter are associated with DM solitons, the method applies directly to the transmission format of quasi-linear return-to-zero modes.[10] The resulting formulas for the frequency and timing shifts are valid for arbitrary channel separations, as is necessary for dense WDM studies.

We begin the analysis with the perturbed nonlinear Schrödinger equation

$$iu_z + (1/2)D(z)u_{tt} + g(z)|u|^2u = 0\,, \qquad (1)$$

written in dimensionless variables, which correspond to physical units that we obtain by taking $t = t_{\text{ref}}/t_*$, $z = z_{\text{lab}}/z_*$, $D = -k''/k_*''$, and $u = E/\sqrt{g(z)P_*}$, where $D(z)$ represents the fiber dispersion, t_{ref} and z_{lab} are the retarded time and propagation distance, respectively, and E denotes the slowly varying envelope of the optical field. Typical numbers used here are $t_* = 10.27$ ps ($\tau_{\text{FWHM}} = 18.1$ ps), $P_* = 1$ mW, $z_* = 400$ km, $k_*'' = 0.264$ ps^2/km, and a DM map length of 40 km. The power variation due to loss and amplification is represented by $g(z) = g_0 \exp(-2\Gamma z)$, $g_0 = 2\Gamma z_a/[1 - \exp(-2\Gamma z_a)]$ on $0 < z < z_a$, $z_a = l_a/z_* = 0.1$, where l_a is the amplifier spacing, which we take to be equal to the DM map length.

We model strong dispersion management by taking $D(z)$ to be a periodic function alternating between constant dispersions with period z_a. By use of a multiscale perturbation analysis, it has been shown that, to leading order, a DM soliton in Fourier space is[11]

$$\hat{u}(z,\omega) = F(\omega,\lambda)\exp[-iC_1(\zeta)\omega^2/2]\exp(i\lambda^2 z/2)\,, \qquad (2)$$

where $C_1(\zeta) = \int_0^\zeta \Delta(\zeta')\mathrm{d}\zeta'$, and the local dispersion is $D(z) = \langle D\rangle + \Delta(z/z_a)/z_a$, where $\langle D\rangle$ is the average dispersion over a period z_a; also, $\Delta(\zeta) = \{\Delta_1{:}0 \le |\zeta| \le \theta/2, \Delta_2{:}\theta/2 \le |\zeta| \le 1/2\}$ is a periodic function of $\zeta = z/z_a$ and the map strength is given by $s = [\Delta_1\theta - \Delta_2(1-\theta)]/4$. In Eq. (2), λ is referred to as the eigenvalue of the DM soliton; it is related to its energy.

In Eq. (2) the pulse shape can be well approximated by a chirped Gaussian, $F(\omega,\lambda) = \alpha\exp[-(\beta + iC_0)\omega^2]$; note that, when loss is ignored, $C_0 = 0$.[11] Using this approximation and noting that Eq. (1) is invariant under a Galilean transform, we can write u_1 and u_2, which are two solutions of Eq. (1) with frequencies Ω_1 and Ω_2, respectively, in physical space as

$$u_1(z,t) = \frac{\alpha}{\{2\pi[\beta + iC(\zeta)\}^{1/2}}\exp\left\{-\frac{[t + \Omega_1\tilde{D}(z)]^2}{2[\beta + iC(\zeta)]}\right\}$$
$$\times\exp\{+(i/2)[\lambda^2 z + 2\Omega_1 t - \Omega_1{}^2\tilde{D}(z)]\}\,, \qquad (3)$$

where $C(\zeta) = C_0 + C_1(\zeta)$, the cumulative dispersion is $\tilde{D}(z) = \int_{z_0}^{z} D(z')\mathrm{d}z' = (z - z_0)\langle D\rangle + C(z/z_a) - C(z_0/z_a)$, and $u_2(z,t)$ is the same but with Ω_1 replaced with Ω_2. We note that z_0 corresponds to the average collision point of the two pulses u_1 and u_2, defined as the point of collision based on the average group velocities $\langle D\rangle\Omega_1$ and $\langle D\rangle\Omega_2$.

From differentiation of the standard equation for average frequency, $\Omega = \mathrm{Im}[\int_{-\infty}^{\infty}(\partial u/\partial t)u^{*}\mathrm{d}t]/W$, with the energy $W_1 = W_2 = W = \int_{-\infty}^{\infty}|u|^2\mathrm{d}t$, it follows from Eq. (1), if we assume that $u \sim u_1 + u_2$ and ignore four-wave mixing terms (which occur in frequency channels that are widely separated from u_1 and u_2), that

$$\frac{\mathrm{d}\Omega_1}{\mathrm{d}z} = \frac{2g(z)}{W_1}\int_{-\infty}^{\infty}|u_1|^2\,\frac{\partial|u_2|^2}{\partial t}\,\mathrm{d}t\,. \tag{4}$$

Equation (4) is independent of the pulse shape. For classical solitons, u_1 and u_2 can be taken to be sech-type pulses,[1] whereas for DM solitons we use Eq. (3). Substituting u_1 and u_2 into Eq. (4) yields

$$\frac{\mathrm{d}\Omega_1}{\mathrm{d}z} = -A(\Omega_1 - \Omega_2)\,\frac{\tilde{D}(z)g(z)}{B^3(\zeta)} \times \exp\left\{-(\beta/2)\left[\frac{\tilde{D}(z)}{B(\zeta)}(\Omega_1 - \Omega_2)\right]^2\right\}, \tag{5}$$

where $A = \alpha^4\sqrt{\beta/2}/(2W\pi^{3/2})$, $B^2(\zeta) = \beta^2 + C^2(\zeta)$, and $\mathrm{d}\Omega_1/\mathrm{d}z = -\mathrm{d}\Omega_2/\mathrm{d}z$. The two-channel system is symmetric, and so we can, without loss of generality, define $\Omega = (\Omega_1 - \Omega_2)/2$.

The residual frequency shift, $\Delta\Omega(z) = \Omega(z) - \Omega(-\infty)$ is obtained by integration of Eq. (5). Assuming that $\Omega(z)$ is a slowly varying function (i.e., $|\mathrm{d}\Omega/\mathrm{d}z| \ll 1$), we can treat Ω on the right-hand side as a constant, obtaining

$$\Delta\Omega(z) = 2A\Omega\int_{-\infty}^{z} g(z)\tilde{D}(z) \times \exp\left\{-2\beta\left[\frac{\tilde{D}(z)}{B(\zeta)}\,\Omega\right]^2\right\}/B^3(\zeta)\mathrm{d}z\,. \tag{6}$$

From Eq. (6) we can obtain the timing shift by integrating $\mathrm{d}\langle t\rangle/\mathrm{d}z = D(z)[\Omega(z) - \Omega(-\infty)] = D(z)\Delta\Omega(z)$, where $\langle t\rangle = \int_{-\infty}^{\infty} t|u|^2\mathrm{d}t$ is the average position of the soliton, to find, when there is a change in the order of integration,

$$\delta t = \int_{-\infty}^{L} D(z)\Delta\Omega(z)\mathrm{d}z = \int_{-\infty}^{L}\frac{\mathrm{d}\Omega}{\mathrm{d}z'}\left(\int_{z'}^{L} D\mathrm{d}z\right)\mathrm{d}z', \tag{7}$$

where L is the length of the fiber. Equation (7) can be split as $\delta t = \tilde{D}(L)\Delta\Omega_{\mathrm{res}} + \delta t_{\mathrm{res}}$, where $\delta\Omega_{\mathrm{res}}$ is the residual frequency shift and

$$\delta t_{\mathrm{res}} = -2A\Omega\int_{-\infty}^{L} g(z)\tilde{D}^2(z) \times \exp\left\{-2\beta\left[\frac{\tilde{D}(z)}{B(\zeta)}\,\Omega\right]^2\right\}/B^3(\zeta)\mathrm{d}z\,, \tag{8}$$

is the residual timing (position) shift. These two contributions to the resultant timing shift were noted in Refs. 4 and 12.

To calculate the frequency or timing shifts, we can numerically integrate Eq. (6) or Eq. (7), but this is computationally intensive. Here we use a semianalytical method to compute both frequency and timing shifts, which is computationally efficient. We transform the integral into a Poisson sum, of which only a few terms usually need to be evaluated. To do this we write Eq. (6) as a sum of integrals over an amplifier spacing, note the periodicity of $C(\zeta)$, and exchange the order of integration and summation to get

$$\begin{aligned}\Delta\Omega_{\mathrm{res}} &= 2A\Omega\sum_{m=-\infty}^{\infty}\int_{mz_a}^{(m+1)z_a}\frac{g(z)\tilde{D}(z)}{B^3(\zeta)} \times \exp\left\{-2\beta\left[\frac{\tilde{D}(z)}{B(\zeta)}\,\Omega\right]^2\right\}\mathrm{d}z \\ &= 2A\Omega\int_0^{z_a} g(z)\sum_{m=-\infty}^{\infty} f[x_m + \tilde{D}(z)]/B^3(\zeta)\mathrm{d}z\,,\end{aligned} \tag{9}$$

where $x_m = mz_a\langle D\rangle$ and $f(x) = x\exp[-2\beta\Omega^2x^2/B^2(\zeta)]$.

We use the Poisson sum formula to rewrite the sum as[13]

$$\sum_{m=-\infty}^{\infty} f(x_m + \tilde{D}) = \frac{1}{z_a\langle D\rangle}\sum_{n=-\infty}^{\infty}\hat{F}\left(\frac{2n\pi}{z_a\langle D\rangle}\right)\exp\left(\frac{2in\pi\tilde{D}}{z_a\langle D\rangle}\right)$$

in terms of the Fourier transform of $f(x)$, $\hat{F}(k) = \int_{-\infty}^{\infty}\exp(-ikx)f(x)\mathrm{d}x = -ikB^3\sqrt{\pi/2}\,\exp[-B^2(\zeta)k^2/(8\beta\Omega^2)]/(4\beta^{3/2}\Omega^3)$. Substituting this into Eq. (9), we obtain the residual frequency shift as a sum in the Fourier domain:

$$\Delta\Omega_{\mathrm{res}}(z_0) = -i\,\frac{\alpha^4}{4W\beta(\Omega z_a\langle D\rangle)^2}\sum_{n=-\infty}^{\infty} n\int_0^{z_a} g(z) \times \exp\left\{-\frac{1}{2\beta}\left[\frac{n\pi B(\zeta)}{z_a\langle D\rangle\Omega}\right]^2 + \frac{2in\pi\tilde{D}(z)}{z_a\langle D\rangle}\right\}\mathrm{d}z\,. \tag{10}$$

It is straightforward to compute the frequency shift from this expression with the integral evaluated numerically only over one amplifier spacing. For moderate frequency separations the exponential decays rapidly in n, so we need to evaluate the sum over only a very small range of n, typically $n = \pm 1, \pm 2$. An analytical formula for the frequency shift can be obtained by integration of Eq. (10), yielding an expression containing error functions. However, the resulting equation is lengthy and will be given elsewhere.

As seen in Eq. (7), the timing shift is composed of two parts: δt_1 contains a contribution from $\tilde{D}(L)$ that grows linearly with system length L and δt_{res} does not contain this growing term.

When δt_{res} is significant, we can also calculate it by use of the Poisson sum method. In the same way as described for the frequency shift, we arrive at an expression for δt_{res}:

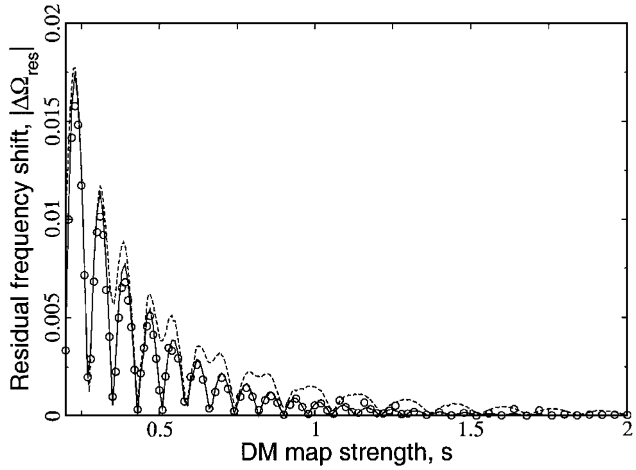

Fig. 1. Maximum value of $\Delta\Omega_{\text{res}}$ versus the DM map strength s, as calculated with the Poisson sum method. Solid curve, $\Gamma = 0$; dashed curve, $\Gamma z_a = 1$; circles, numerical simulations with $\Gamma = 0$. Here $\Omega = 12$ and $W = 2.5$ (nondimensional units).

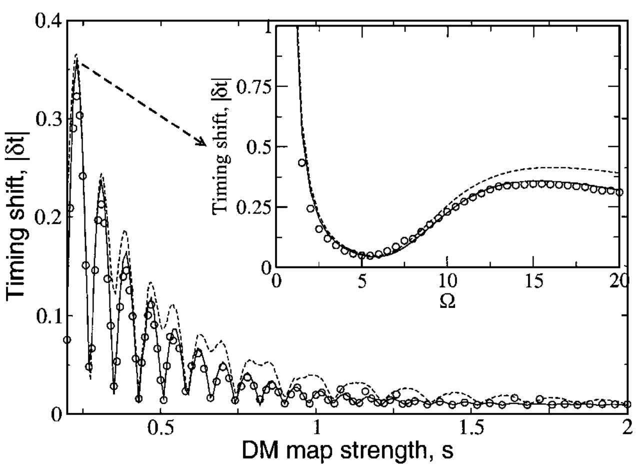

Fig. 2. Maximum value of δt versus the DM map strength s, as calculated with the Poisson sum method. Solid curve, $|\delta t|$ for $\Gamma = 0$; dashed curve, $|\delta t|$ $\Gamma z_a = 1$; circles $|\delta t|$ from numerical simulations with $\Gamma = 0$. Here $\Omega = 12$, $W = 2.5$, and $L = 20$ (in nondimensional units). Inset, $|\delta t|$ versus Ω for $s = 0.23$.

$$\delta t_{\text{res}}(z_0) = \frac{\alpha^4}{8W\beta\pi z_a\langle D\rangle\Omega^2}\sum_{n=-\infty}^{\infty}\int_0^{z_a} g(z) \times\left[\frac{B^2(\zeta)}{\beta}\left(\frac{n\pi}{z_a\langle D\rangle\Omega}\right)^2 - 1\right] \times \exp\left[-\frac{1}{2\beta}\left(\frac{n\pi B(\zeta)}{z_a\langle D\rangle\Omega}\right)^2 + i\,\frac{2n\pi\tilde{D}(z)}{z_a\langle D\rangle}\right]\mathrm{d}z\,. \tag{11}$$

As with the frequency shift, an analytical form for Eq. (11) can be found. We also note that as $\langle D\rangle \to 0$ both the residual frequency and timing shifts grow without bound. This growth puts a limit on how small $\langle D\rangle$ can be.

In calculating the results that follow, we used a pulse shape given by Eq. (3) with a fixed energy of $W = 2.5$. Throughout, we take $\langle D\rangle = 1$ and $\theta = 1/2$. The results for the Poisson sum method are calculated numerically from Eqs. (10) and (11). We calculate the maximum of the frequency or timing shifts over a range of collision points, noting that $\Delta\Omega$ is periodic in z_0 with period z_a.

Figure 1 shows the residual frequency shift as a function of the map strength, where we take $\Omega = 12$ corresponding to a pulse spacing of 372 GHz, without loss $\Gamma = 0$ and with loss $\Gamma z_a = 1$ (corresponding to a loss of 0.2 dB/km). Results obtained from the numerical simulation of Eq. (1) for the lossless case are shown and are in evident agreement. Figure 2 shows the timing shift, δt, with respect to the DM map strength, with $\Omega = 12$ and $L = 20$, corresponding to a system length of 8000 km.

In conclusion, using a Gaussian approximation to the DM soliton pulse profile with the Poisson sum technique has allowed us to obtain formulas that can be readily computed; analytical representations for all quantities can be obtained. It was shown that the residual frequency shift decreases rapidly with map strength. The timing shift is composed of two terms: One is effectively the residual frequency shift multiplied by the system length, and the other is the residual timing shift. The residual timing shift, δt_{res}, is the dominant term in the timing shift for larger map strength and for small Ω. This method applies to quasi-linear return-to-zero pulses and can be extended to the study of WDM systems and partial collisions.

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